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LOCAL SPECTRAL PROPERTIES OF QUASI-DECOMPOSABLE OPERATORS

Jong-Kwang Yoo* and Heung Joon Oh**

Abstract. In this paper we investigate the local spectral properties of quasidecomposable operators. We show that if $T \in L(X)$ is quasi-decomposable, then T has the weak-SDP and $\sigma_{loc}(T) = \sigma(T)$. Also, we show that the quasi-decomposability is preserved under commuting quasi-nilpotent perturbations. Moreover, we show that if $f : U \to \mathbb{C}$ is an analytic and injective on an open neighborhood U of $\sigma(T)$, then $T \in L(X)$ is quasi-decomposable if and only if $f(T)$ is quasi-decomposable. Finally, if $T \in L(X)$ and $S \in L(Y)$ are asymptotically similar, then T is quasi-decomposable if and only if S does.

1. Introduction and basic definitions

Let X and Y be complex Banach spaces over the complex field $\mathbb{C}$, and let $L(X, Y)$ be the Banach algebra of all bounded linear operators from X to Y, and let $L(X) := L(X, X)$. Given $T \in L(X)$, we use $\sigma(T)$, $\sigma_p(T)$, $\sigma_{sur}(T)$, $\sigma_{ap}(T)$, and $\rho(T)$ to denote the spectrum, the point spectrum, the surjectivity spectrum, the approximate point spectrum, and the resolvent set of T, respectively. As usual, given $T \in L(X)$, let $kerT$ and $T(X)$ stand for the kernel and range of T. Let $Lat(T)$ stand for the collection of all T-invariant closed linear subspaces of X. For $Y \in Lat(T)$, let $T|Y$ denote the operator given by the restriction of T to Y.

The *local resolvent set* $\rho_T(x)$ of an operator T at a point $x \in X$ is the union of all open subsets U of $\mathbb{C}$ for which there is an analytic function $f : U \to X$ that satisfies the equation

$$(\lambda I - T)f(\lambda) = x \text{ for all } \lambda \in U.$$

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Correspondence should be addressed to Jong-Kwang Yoo, jkyoo@cdu.ac.kr.

The *local spectrum* of T at x is defined by $\sigma_T(x) := \mathbb{C} \setminus \rho_T(x)$, and obviously $\sigma_T(x)$ is a closed subset of $\sigma(T)$.

An operator $T \in L(X)$ is said to have the *single-valued extension property* at $\lambda \in \mathbb{C}$ (abbreviated SVEP at λ), if for every open disc U centered at λ, the only analytic function $f : U \to X$ which satisfies the equation

$$(\mu I - T)f(\mu) = 0 \ \text{ for all } \mu \in U$$

is the constant function $f \equiv 0$. An operator $T \in L(X)$ is said to have the SVEP if T has the SVEP at every point $\lambda \in \mathbb{C}$.

For every subset F of $\mathbb{C}$, the *analytic spectral subspace* of T associated with F is the set $X_T(F) := \{x \in X : \sigma_T(x) \subseteq F\}$. It is easy to see from definition that $X_T(F)$ is a $T-$invariant linear subspace of X and that $X_T(F_1) \subseteq X_T(F_2)$ whenever $F_1 \subseteq F_2$. It is well known from Proposition 1.2.16 of [11] that

$$T \text{ has SVEP} \quad \Leftrightarrow \quad X_T(\phi) = \{0\} \quad \Leftrightarrow \quad X_T(\phi) \ \text{ is closed.}$$

It is more appropriate to work with another kind of spectral subspaces: for each closed set $F \subseteq \mathbb{C}$, the *glocal spectral subspace* $\mathcal{X}_T(F)$ consists of all $x \in X$ for which there exists an analytic function $f : \mathbb{C} \setminus F \to X$ such that $(\lambda I - T)f(\lambda) = x$ for each $\lambda \in \mathbb{C} \setminus F$. It is clear that $\mathcal{X}_T(F) \subseteq X_T(F)$. Evidently, by Proposition 3.3.2 of [11], T has SVEP if and only if $\mathcal{X}_T(F) = X_T(F)$ for all closed sets $F \subseteq \mathbb{C}$. We emphasize that neither the local nor the glocal spectral subspaces have to be closed. These linear subspaces play a fundamental role in the spectral theory of operators on Banach spaces.

An operator $T \in L(X)$ on a complex Banach space X is said to have *Dunford's property (C)* (shortly, property (C)) if $X_T(F)$ is closed for every closed set $F \subseteq \mathbb{C}$.

Recall from [11] that an operator $T \in L(X)$ is said to have *Bishop's property* (β) if, for every open subset U of $\mathbb{C}$ and for every sequence of analytic functions $f_n : U \to X$ for which $(\lambda I - T)f_n(\lambda)$ converges uniformly to zero on each compact subset of U, it follows that also $f_n(\lambda) \to 0$ as $n \to \infty$, locally uniformly on U. An operator $T \in L(X)$ is said to have the *decompositions property* (δ) if for each open cover $\{U_1, U_2\}$ of $\mathbb{C}$ and for each $x \in X$ there are a pair of elements $u_i \in X$ and a pair of analytic functions $f_i : \mathbb{C} \setminus \overline{U_i} \longrightarrow X$ such that $x = u_1 + u_2$, $u_i = (\lambda I - T)f_i(\lambda)$ for all $\lambda \in \mathbb{C} \setminus \overline{U_i}$, $(i = 1, 2)$.

Properties (β) and (δ) are known to be dual to each other in the sense that $T \in L(X)$ has (β) if and only if $T^* \in L(X^*)$ satisfies (δ), where T^* is the adjoint operator on the dual space X^*. It is clear that the decompositions property (δ) is inherited by quotients and decomposition property (δ) means precisely that $X = \mathcal{X}_T(\overline{U_1}) + \mathcal{X}_T(\overline{U_2})$ for every open covering $\{U_1, U_2\}$ of $\mathbb{C}$, see [1] and [11].

Recall from [1] that an operator $T \in L(X)$ is called *decomposable* if, for every open covering $\{U_1, U_2\}$ of the complex plane $\mathbb{C}$, there exist $Y_i \in Lat(T)$ such that

$$X = Y_1 + Y_2, \quad \text{and} \quad \sigma(T|Y_i) \subseteq U_i \quad \text{for all} \quad i = 1, \cdots, n.$$

Clearly, $Y_i \subseteq X_T(\overline{U_i})$ for all $i = 1, \cdots, n$, it is easily shown that if $T \in L(X)$ is decomposable, then $X = X_T(\overline{U_1}) + X_T(\overline{U_2})$ for every open cover $\{U_1, U_2\}$ of $\mathbb{C}$.

A weaker version of decomposable operators is given by operators that have *weak spectral decomposition property*, abbreviated *weak-SDP* provided that there exist $Y_i \in Lat(T)$ such that

$$X = \overline{Y_1 + Y_2 + \cdots + Y_n} \quad \text{and} \quad \sigma(T|Y_i) \subseteq U_i \quad \text{for all} \quad i = 1, \cdots, n.$$

Evidently, $Y_i \subseteq X_T(\overline{U_i})$ for all $i = 1, \cdots, n$, and it is easily shown that if $T \in L(X)$ has the weak-SDP, then

$$X = \overline{X_T(\overline{U_1}) + X_T(\overline{U_2}) + \cdots + X_T(\overline{U_n})}$$

for every open cover $\{U_1, \cdots, U_n\}$ of $\mathbb{C}$. In [4], E. Albrecht gives an example that shows that the class of bounded linear operators with weak-SDP contains strictly the class of decomposable operators. Note that it follows from [1] and [4] that operators with the weak 2-SDP need not have the property (δ), and there are operators with the property (δ) with no weak 2-SDP.

2. Results

Definition 2.1. An operator $T \in L(X)$ is quasi-decomposable if T has property (C) and for every finite open cover $\{U_1, \cdots, U_n\}$ of $\mathbb{C}$, the sum

$$X_T(\overline{U_1}) + \cdots + X_T(\overline{U_n})$$

is dense in X.

The class of quasi-decomposable operators contains all normal operators and more generally all spectral operators. Operators with totally disconnected spectrum are quasi-decomposable by the Riesz functional calculus. In particular, compact and algebraic operators are quasi-decomposable. It is clear that every quasi-decomposable operator has SVEP. Evidently, every decomposable operators are quasi-decomposable, but an example due to Albrecht illustrates that the converse is not true in general, see [4]. If T is the unilateral left shift on the sequence space $X = \ell^p(\mathbb{N})$ for arbitrary $1 \leq p < \infty$, then T does not have the SVEP, and hence T is not quasi-decomposable.

Theorem 2.2. *If $T \in L(X)$ is quasi-decomposable then T has the weak-SDP. Moreover, for every open cover $\{U_1, \cdots, U_n\}$ of $\sigma(T)$, there exist $Y_i \in Lat(T)$ such that*

(a) $X = \overline{Y_1 + Y_2 + \cdots + Y_n}$;
(b) $\sigma(T|Y_i) \subseteq U_i$ *for all* $i = 1, \cdots, n$;
(c) $\sigma(T) = \bigcup_{i=1}^n \sigma(T|Y_i)$.

Proof. Let $\{U_1, \cdots, U_n\}$ be an open cover of $\mathbb{C}$. We can choose an open cover $\{V_1, \cdots, V_n\}$ of $\mathbb{C}$ such that $V_i \subseteq \overline{V_i} \subseteq U_i$ for all $i = 1, \cdots, n$. Since T is quasi–decomposable, the sum

$$X_T(\overline{V_1}) + \cdots + X_T(\overline{V_n})$$

is dense in X. Since T has property (C), $X_T(\overline{V_i})$ is closed. Let $Y_i := X_T(\overline{V_i}) \in Lat(T)$. It follows from Proposition 1.2.20 of [11] that

$$\sigma(T|Y_i) = \sigma(T|X_T(\overline{V_i})) \subseteq \overline{V_i} \subseteq U_i$$

for all $i = 1, \cdots n$. Since $Y_i \subseteq X_T(\overline{V_i})$ for all $i = 1, \cdots, n$, we obtain

$$X = \overline{Y_1 + Y_2 + \cdots + Y_n},$$

and hence T has the weak SDP. Let $K := \bigcup_{i=1}^n \sigma(T|Y_i)$. Suppose on the contary that K is a proper subset of $\sigma(T)$. Then $X_T(K)$ is a proper subspace of $X = X_T(\sigma(T))$ and $Y_i \subseteq X_T(K)$ for all $i = 1, \cdots, n$. This implies that $X \subseteq X_T(K)$, contradiction. Hence $\sigma(T) = \cup_{i=1}^n \sigma(T|Y_i)$. □

Theorem 2.3. *If $T \in L(X)$ is quasi-decomposable, then T^* has SVEP. Moreover, $\sigma(T) = \sigma_{ap}(T)$.*

Proof. It follows from Proposition 2.5.1 of [11] that

$$\mathcal{X}^*_{T^*}(\phi) \subseteq \mathcal{X}_T(\mathbb{C})^\perp = X_T(\mathbb{C})^\perp = \{0\}.$$

By Proposition 1.2.16 (f) of [11], T^* has SVEP. It follows from Theorem 2.42 of [1] that $\sigma(T) = \sigma(T^*) = \sigma_{sur}(T^*) = \sigma_{ap}(T)$. □

Recall from [10] that an operator $A \in L(X,Y)$ is said to *intertwine* $S \in L(Y)$ and $T \in L(X)$ *asymptotically* if

$$\|C(S,T)^n(A)\|^{\frac{1}{n}} \to 0 \ \text{ as } \ n \to \infty,$$

where $C(S,T) : L(X,Y) \to L(X,Y)$ is defined by $C(S,T)(A) := SA - AT$ for all $A \in L(X,Y)$ and $C(S,T)^n(A) := C(S,T)^{n-1}(SA - AT)$ for all $n \in \mathbb{N}$.

THEOREM 2.4. *Let $Q \in L(X)$ be a quasi-nilpotent commuting with $T \in L(X)$. Then T is quasi-decomposable if and only if $T + Q$ is quasi-decomposable.*

Proof. We show that $X_T(F) = X_{T+Q}(F)$ for all closed subsets F of $\mathbb{C}$. Since $TQ = QT$, we have

$$C(T+Q,T)^n(I) = Q^n \ \text{ and } \ C(T,T+Q)^n(I) = (-1)^n Q^n$$

for all $n \in \mathbb{N}$. Since Q is quasi-nilpotent, we have

$$\lim_{n\to\infty} \|C(T+Q,T)^n(I)\|^{\frac{1}{n}} = \lim_{n\to\infty} \|C(T,T+Q)^n(I)\|^{\frac{1}{n}} = 0.$$

We follow the line of reasoning in the proof of Theorem 2.3.3 in [7]. We show that $\sigma_{T+Q}(x) \subseteq \sigma_T(x)$ for all $x \in X$. Let $x \in X$ and let $\lambda_0 \notin \sigma_T(x)$. Then there exists an analytic function $f : U \to X$ on an open neighborhood U of λ_0 such that

$$(\mu I - T)f(\mu) = x \ \text{ for all } \mu \in U.$$

Choose two closed discs V, W centered at λ_0 with radii $0 < s < r$ such that $V \subseteq W \subseteq U$. Since $f(W)$ is compact, there exists a constant $M \geq 0$ such that

$$\|f(\lambda)\| \leq M \ \text{ for all } \lambda \in W.$$

For each $\lambda \in V$, we obtain from Cauchy's integral formula that

$$\|\frac{f^{(n)}(\lambda)}{n!}\| = \|\frac{1}{2\pi i}\int_{\partial W} \frac{f(\mu)}{(\mu-\lambda)^{n+1}}d\mu\| \leq \frac{Mr}{(r-s)^{n+1}}$$

for all $n = 0, 1, \cdots$. Let $\epsilon := (r-s)/2$. Since $\lim_{n\to\infty} \|C(T+Q,T)^n(I)\|^{\frac{1}{n}} = 0$, there exists $K \geq 0$ such that

$$\|C(T+Q,T)^n(I)\|^{\frac{1}{n}} \leq K\epsilon^n$$

for all $n = 0, 1, \cdots$. Thus we have

$$\|\|C(T+Q,T)^n(I)\frac{f^{(n)}(\lambda)}{n!}\| \leq \frac{MKr}{2^n(r-s)}$$

for all $\lambda \in V$ and $n = 0, 1, \cdots$. We define $g : U \to X$ by

$$g(\lambda) := \sum_{n=0}^{\infty} C(T+Q,T)^n(I)\frac{f^{(n)}(\lambda)}{n!} \quad \text{for all} \quad \lambda \in U.$$

Then $g(\lambda)$ converges locally uniformly on V and hence locally uniformly on U. Since $(\lambda I - T)f(\lambda) = x$ for all $\lambda \in U$, we obtain by induction that

$$(\lambda I - T)f^{(n)}(\lambda) = nf^{(n-1)}(\lambda)$$

for all $\lambda \in U$ and for all $n \in \mathbb{N}$. Since $C(T+Q,T)^{n+1}(I) = (T+Q)C(T+Q,T)^n(I) - C(T+Q,T)^n(I)T$ for all $n = 0, 1, \cdots$, it is easy to see that

$$(\lambda I - (T+Q))g(\lambda) = A(\lambda I - T)f(\lambda) = Ax$$

for all $\lambda \in U$, which implies that $\lambda \notin \sigma_{T+Q}(x)$. We have proved that

$$\sigma_{T+Q}(x) \subseteq \sigma_T(x) \quad \text{for all } x \in X.$$

The opposite inclusion can be proved in a similar way, and hence $\sigma_{T+Q}(x) = \sigma_T(x)$ for all $x \in X$. Hence $X_{T+Q}(F) = X_T(F)$ for all closed subsets F of $\mathbb{C}$. This implies $T+Q$ has property (C) if and only if T does. For each finite open cover $\{U_1, \cdots, U_n\}$ of $\mathbb{C}$, $X_{T+Q}(\overline{U_i}) = X_T(\overline{U_i})$ for all $i = 1, \cdots, n$. It follows that $T+Q$ is quasi-decomposable if and only if T does. □

For an arbitrary operator $T \in L(X)$ and analytic function $f : U \to \mathbb{C}$ on an open neighborhood U of $\sigma(T)$, let $f(T) \in L(X)$ denote the operator given by the Riesz functional calculus

$$f(T) := \frac{1}{2\pi i}\int_{\Gamma} f(\lambda)(\lambda I - T)^{-1} d\lambda,$$

where Γ is a contour in U that surrounds $\sigma(T)$. The classical spectral mapping theorem asserts that $\sigma(f(T)) = f(\sigma(T))$, see [1] and [11].

Lemma 2.5. ([11]) *Let $T \in L(X)$ and let $f : U \to \mathbb{C}$ be an analytic function on an open neighborhood U of $\sigma(T)$. Then $\mathcal{X}_{f(T)}(F) = \mathcal{X}_T(f^{-1}(F))$ for every closed subset F of $\mathbb{C}$.*

Theorem 2.6. *Let $T \in L(X)$ and let $f : U \to \mathbb{C}$ be analytic and injective on an open neighborhood U of $\sigma(T)$. Then T is quasi-decomposable if and only if $f(T)$ is quasi-decomposable.*

Proof. It follows from the spectral mapping theorem that $f(\sigma(T)) = \sigma(f(T))$. Suppose that T is quasi-decomposable. Then T has property (C). Since $X_{f(T)}(F) = X_T(f^{-1}(F))$ for all closed $F \subseteq \mathbb{C}$, $f(T)$ has property (C). Let $\{U_1, \cdots, U_n\}$ be an open cover of $\sigma(f(T)) = f(\sigma(T))$. Then $\{f^{-1}(U_1), \cdots, f^{-1}(U_n)\}$ is an open cover of $\sigma(T)$. The sum

$$X_T(\overline{f^{-1}(U_1)}) + \cdots + X_T(\overline{f^{-1}(U_n)})$$

is dense in X. Also, it follows from Lemma 2.5 that

$$X_T(\overline{f^{-1}(U_i)}) \subseteq X_T(f^{-1}(\overline{U_i})) = X_{f(T)}(\overline{U_i})$$

for all $i = 1, \cdots, n$, and hence the sum

$$X_{f(T)}(\overline{U_1}) + \cdots + X_{f(T)}(\overline{U_n})$$

is dense in X, which implies that $f(T)$ is quasi-decomposable.

To prove the reverse implication, assume that $f(T)$ is quasi-decomposable. It follows from Lemma 2.5 that T has property (C). Let $\{V_1, \cdots, V_n\}$ be an open cover of $\sigma(T)$, and let $W_i := U \cap V_i (i = 1, \cdots, n)$. Then clearly, $\{W_1, \cdots, W_n\}$ is an open cover of $\sigma(T)$. By the open mapping theorem,

$$\{f(W_1), \cdots, f(W_n)\}$$

is an open cover of $f(\sigma(T)) = \sigma(f(T))$. Thus the sum

$$X_{f(T)}(\overline{f(W_1)}) + \cdots + X_{f(T)}(\overline{f(W_n)})$$

is dense in X. Since f is injective, $X_{f(T)}(\overline{f(W_i)}) = X_T(\overline{W_i})$ for all $i = 1, \cdots, n$. Thus $X_T(\overline{W_1}) + \cdots + X_T(\overline{W_n})$ is dense in X, and hence

$$X_T(\overline{V_1}) + \cdots + X_T(\overline{V_n})$$

is dense in X, which implies that T is quasi-decomposable. □

The operators $T \in L(X)$ and $S \in L(Y)$ are *asymptotically similar* if there is a bijection $A \in L(X, Y)$ such that A intertwines S and T asymptotically and its inverse A^{-1} intertwines T and S asymptotically.

It is well known that if $T \in L(X)$ and $S \in L(Y)$ are asymptotically similar and a corresponding bijection is $A \in L(X, Y)$ for the asymptotic intertwining of (S, T) and (T, S), then $\sigma(T) = \sigma(S)$, $AX_T(F) = Y_S(F)$ and $A^{-1}Y_S(F) = X_T(F)$ for all closed subsets F of $\mathbb{C}$. We show that the quasi-decomposability is preserved under commuting quasi-nilpotent perturbations.

THEOREM 2.7. *Let $T \in L(X)$ and $S \in L(Y)$. Suppose that T and S are asymptotically similar. Then T is quasi-decomposable if and only if S does.*

Proof. Let $A \in L(X, Y)$ be a bijection such that A intertwines S and T asymptotically and its inverse A^{-1} intertwines T and S asymptotically. Then it is easily shown that $\sigma(T) = \sigma(S)$, $AX_T(F) = Y_S(F)$ and $A^{-1}Y_S(F) = X_T(F)$ for all closed subset F of $\mathbb{C}$. This shows that property (C) carries over from T to S. Suppose that T is quasi-decomposable. For any open cover $\{U_1, \cdots, U_n\}$ of $\sigma(T) = \sigma(S)$, the sum

$$X_T(\overline{U_1}) + \cdots + X_T(\overline{U_n})$$

is dense in X. Thus we have

$$Y = A(X) \subseteq \overline{A(X_T(\overline{U_1})) + \cdots + A(X_T(\overline{U_n}))} = \overline{Y_S(\overline{U_1}) + \cdots + Y_S(\overline{U_n})},$$

which implies that the sum

$$Y_S(\overline{U_1}) + \cdots + Y_S(\overline{U_n})$$

is dense in Y, and hence S is quasi-decomposable. The reverse implication is similar. This completes the proof. $\square$

The *localizable spectrum* $\sigma_{loc}(T)$ of an operator $T \in L(X)$ is defined as a set of all $\lambda \in \mathbb{C}$ for which $X_T(\overline{V}) \neq \{0\}$ for every open neighborhood V of λ.

It is well known that $\sigma_{loc}(T)$ is a closed subset of $\sigma(T)$ and that $\sigma_{loc}(T)$ contains the point spectrum $\sigma_p(T)$ and is included in the approximate point spectrum $\sigma_{ap}(T)$ of T. It is clear that if T does not have the SVEP, then $\sigma_{loc}(T) = \sigma(T)$, since $X_T(\phi) \subseteq X_T(\overline{U})$ for every open neighborhood U of $\lambda \in \mathbb{C}$, see more details [8], [12], [13] and [14]. As shown in [13], the localizable spectrum plays an important role in the theory of invariant subspaces.

THEOREM 2.8. *Suppose that $T \in L(X)$ is quasi-decomposable. Then $\sigma_{loc}(T) = \sigma_{ap}(T) = \sigma(T)$.*

Proof. Obviously, $\sigma_{ap}(T) \subseteq \sigma(T)$. To verify that $\sigma_{loc}(T) \subseteq \sigma_{ap}(T)$, we assume that $\lambda \notin \sigma_{ap}(T)$. Then there exists some constant $r > 0$ such that $\overline{V} \cap \sigma_{ap}(T) = \phi$, where V is an open disc centered at λ with radius r. By Theorem 3.3.12 (d) of [11], $X_T(\overline{V}) = \mathcal{X}_T(\overline{V}) = \{0\}$. This implies that $\lambda \notin \sigma_{loc}(T)$, and hence $\sigma_{loc}(T) \subseteq \sigma_{ap}(T)$. Finally, we show that $\sigma_{ap}(T) \subseteq \sigma_{loc}(T)$. Let $\lambda \notin \sigma_{loc}(T)$. Then there exists $\epsilon > 0$ such that $X_T(\overline{U_1}) = \{0\}$, where U_1 is an open disc centered at λ with radius ϵ. We consider $U_2 := \mathbb{C} \setminus \overline{W}$, where W is an open disc centered at λ with radius $\epsilon/2$. Since $\{U_1, U_2\}$ is an open cover of $\mathbb{C}$, we choose an open cover

$\{W_1, W_2\}$ of $\mathbb{C}$ such that $W_i \subseteq \overline{W_i} \subseteq U_i$ for all $i = 1, 2$. By Theorem 2.2, T has the weak-SDP. Thus there exist $Y_i \in Lat(T)$ such that

$$X = \overline{Y_1 + Y_2}, \quad \text{and} \quad \sigma(T|Y_i) \subseteq W_i \quad \text{for all } i = 1, 2.$$

This implies that

$$Y_1 \subseteq X_T(\overline{W_1}) \subseteq X_T(\overline{U}) = \{0\},$$

and hence $X = Y_2$. Thus $\sigma(T) = \sigma(T|Y_2) \subseteq W_2$. Since $\lambda \notin U_2$, we obtain $\lambda \notin \sigma(T)$, which implies that $\sigma(T) \subseteq \sigma_{loc}(T)$. □

It is well known that if $T \in L(X)$, $S \in L(Y)$, $X_1 \in Lat(T)$ and $Y_1 \in Lat(S)$, then we have

$$\sigma(T \oplus S|X_1 \oplus Y_1) = \sigma(T|X_1) \cup \sigma(S|Y_1),$$

where $X_1 \oplus Y_1$ is considered as a subspace of $X \oplus Y := \{x \oplus y : x \in X \text{ and } y \in Y\}$ and $\|x \oplus y\| = (\|x\|^2 + \|y\|^2)^{1/2}$.

The following lemma is an immediate cosequence of Proposition 1.4 of [7].

LEMMA 2.9. ([7]) *Let $T \in L(X)$ and $S \in L(Y)$. Then $(X \oplus Y)_{T \oplus S}(F) = X_T(F) \oplus Y_S(F)$ for all subsets F of $\mathbb{C}$.*

THEOREM 2.10. *Let $T \in L(X)$ and $S \in L(Y)$. If $T \in L(X)$ and $S \in L(Y)$ are quasi-decomposable then $T \oplus S \in L(X \oplus Y)$ is also quasi-decomposable.*

Proof. Suppose that $T \in L(X)$ and $S \in L(Y)$ are quasi-decomposable. By Lemma 2.9, for each closed $F \subseteq \mathbb{C}$, $(X \oplus Y)_{T \oplus S}(F)$ is closed. Thus $T \oplus S$ has property (C). Let $\{U_1, \cdots, U_n\}$ be an open cover of $\sigma(T \oplus S) = \sigma(T) \cup \sigma(S)$. Then the sum

$$X_T(\overline{U_1}) + \cdots + X_T(\overline{U_n})$$

is dense in X and the sum

$$Y_S(\overline{U_1}) + \cdots + Y_S(\overline{U_n})$$

is dense in Y. Since $(X \oplus Y)_{T \oplus S}(\overline{U_i}) = X_T(\overline{U_i}) \oplus Y_S(\overline{U_i})$ for all $i = 1, \cdots, n$,

$$(X \oplus Y)_{T \oplus S}(\overline{U_1}) + \cdots + (X \oplus Y)_{T \oplus S}(\overline{U_n}) = (\sum_{i=1}^{n} X_T(\overline{U_i})) \oplus (\sum_{i=1}^{n} Y_S(\overline{U_i}))$$

is dense in $X \oplus Y$. Hence $T \oplus S$ is quasi-decomposable. □

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Department of Liberal Arts and Science
Chodang University
Muan 534-701, Republic of Korea
E-mail: jkyoo@cdu.ac.kr

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Department of Liberal Arts and Science
Chodang University
Muan 534-701, Republic of Korea
E-mail: hjoh@cdu.ac.kr